

Mathematical Modelling and Numerical Simulation with Applications

Volume 6 | Issue 1

Article 9

2026

Measuring market risk through Entropic VaR

Dragomir Nedeltchev

Bulgarian Academy of Sciences, Bulgaria, dnedeltchev@math.bas.bg

Tsvetelin Zaevski

Centre of Excellence in Informatics and Information and Communication Technologies, Bulgaria,

t_s_zaevski@math.bas.bg

Follow this and additional works at: <https://mmnsa.researchcommons.org/journal>



Part of the [Dynamic Systems Commons](#), [Finance Commons](#), [Numerical Analysis and Computation Commons](#), and the [Statistical Models Commons](#)

Recommended Citation

Nedeltchev, Dragomir and Zaevski, Tsvetelin (2026) "Measuring market risk through Entropic VaR," *Mathematical Modelling and Numerical Simulation with Applications*: Vol. 6: Iss. 1, Article 9.

DOI: <https://doi.org/10.53391/2791-8564.1024>

Available at: <https://mmnsa.researchcommons.org/journal/vol6/iss1/9>

This Original Research Article is brought to you for free and open access by Mathematical Modelling and Numerical Simulation with Applications. It has been accepted for inclusion in Mathematical Modelling and Numerical Simulation with Applications by an authorized editor of Mathematical Modelling and Numerical Simulation with Applications.



Digital Commons™



Mathematical Modelling and Numerical Simulation with Applications

2026, 6(1), 251–278

<https://mmnsa.researchcommons.org>

ISSN Online: 2791-8564 / Open Access

<https://doi.org/10.53391/2791-8564.1024>


RESEARCH PAPER

Measuring market risk through Entropic VaR

Dragomir Nedeltchev ^{1,†} and Tsvetelin Zaeovski ^{1,2,*,‡}

¹Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria

²Centre of Excellence in Informatics and Information and Communication Technologies, Sofia, Bulgaria

* Corresponding Author

† dnedeltchev@math.bas.bg, dnedeltchev65@yahoo.com (Dragomir Nedeltchev); t_s_zaeovski@math.bas.bg, t_s_zaeovski@abv.bg (Tsvetelin Zaeovski)

Abstract

The article aims to measure the market risk beyond the basic risk measures like the Value-at-Risk (VaR) and the Expected Shortfall (ES). The Entropic Value-at-Risk is selected among the available measures based on its advantages – it is the coherent upper bound of the VaR and ES. This risk measure is applied to the classical Black-Scholes model as well as to some more realistic ones, such as the exponential tempered stable model (the log-returns are presented by a tempered stable Lévy process), the stochastic volatility model of Heston, its jump extension of Bates, and another stochastic volatility model but with tempered stable jump behavior. Closed or semi-closed form formulas are obtained for all these models. The derived theoretical results are applied to a historical sample for the S&P 500 index. There are several advantages of the use of a stochastic model instead of the direct application of the risk measure to the statistical data. The main advantage is that an appropriate model can capture the inherent features of the financial asset that are hidden in the time series due to an insufficient number of observations. This is especially true for the distribution tails – terms closely related to the risk measures.

Keywords: Entropy risk measure; Entropic VaR; stochastic models; risk-management

AMS 2020 Classification: 60E07; 60E10; 62P05; 91B05; 91G70; 91G10

1 Introduction

Risk measures appear on the agenda of financial mathematicians for two main reasons: providing capital buffers against crises and optimizing portfolios by minimizing their market risk exposure. Regulatory entities issue obligatory recommendations for the credit institutions regarding the capital reserves based on the regulator's acceptance set. Such an approach enables the application of multiple ways to measure the risk. Hence, the question arises as to what the most appropriate

► Received: 12.10.2025 ► Revised: 25.03.2026 ► Accepted: 28.03.2026 ► Published: 31.03.2026

risk measure is and, as a consequence, the most conservative one among the various available choices. Risk managers are required to be aware of the ultimate scenario that a portfolio should be ready to face. Nowadays, the research on Value-at-Risk (VaR) and the Expected Shortfall (ES) remains a high priority for the academic community. Machine learning approaches based on *Mogriifier* recurrent neural networks are applied to improve the VaR/ES estimation accuracy in [1]. On the other hand, a better estimate of VaR and ES necessitates leveraging different novel sophisticated techniques, for example, quantum computing – see [2] and [3]. Lévy-VaR and ES are used as an alternative to the generalized Pareto VaR and ES, to better comprehend the tail behavior – see [4]. [5] proposes to compute the VaR and ES by considering the possible interactions with other observable risks. Raising the ability to forecast VaR and ES is another domain of contemporary research. A better forecast is possible for Chinese stock markets when normal and non-normal GARCH-type filters are combined with the extreme value theory – see [6]. On the other hand, [7] combines the extreme value theory with GARCH-mixed data sampling to forecast the VaR on the crude oil market. VaR/ES forecast improvement is possible when applying machine learning methods like Recurrent Neural Networks and Feed-Forward Neural Networks – see [8]). There is evidence that the dynamic Gerber model results in better VaR/ES forecasts than alternative parametric, non-parametric, and semi-parametric methods for estimating the covariance matrix of returns – see [9]. Furthermore, there is a heightened current interest in VaR/ES research for cryptocurrency markets – see [10]. Also, the portfolio selection and optimization remain a preferred area of researching the VaR/ES during the last couple of years (see, for example [11]). A study in this field based on the expectiles can be found in [12].

On the other hand, the risk managers need a non-intuitive understanding of the upper bound of the market risk that is not too computationally intensive. Being the upper bound of the Value-at-Risk, the Entropic VaR (EVaR) marks the most conservative approach to capital reserves. Additionally, it is a coherent risk measure – see [13] – i.e., it is translation invariant, subadditive, monotone, and positive homogeneous. Another feature of the EVaR is its computational efficiency compared to other sophisticated risk measures, like the Expected Shortfall.

Two streams mark the genesis of the topic this article deals with: the invention of the Entropic Risk Measure and its adoption in the financial area, and the introduction of the EVaR in the field of operations research. [14] introduced the entropic risk measure and later studied its properties in [15]. [16] elaborated deviation measures and the so-called risk envelopes. [17] further developed the EVaR concept. Special attention is paid to studying the coherence and the convexity of the new risk measure in [18] and [19]. This theoretical field was expanded further by introducing new risk measures based on entropy. [20] elaborated the iso-entropic risk measure. [21] introduced the so-called cumulative risk measure. The Weighted Entropic Risk Measure that is consistent with the second-order stochastic dominance and additive for sums of independent random variables is considered in [22]. [23] discussed the risk measures based on some generalizations of the entropy concept: information entropy, cumulative residual entropy, fuzzy entropy, credibility entropy, sine entropy, and hybrid entropy. Generally speaking, the portfolio optimization is the dominant area of applying the entropic risk measures - see ([24]).

The EVaR was introduced by [25] and [26] as a measure based on the generalized relative entropy where $\ln(\cdot)$ plays the role of a convex function. [27] extends the measure as he used Kullback-Leibler divergence for the Kusuoka representation of the EVaR. [28] proved that contrary to the VaR and the ES, the EVaR is strongly monotone over its domain and strictly monotone over a broad sub-domain including all continuous distributions. [29] adjust the new measure to the market realities by adding a Rayleigh distribution to the normal one. [30] elaborated a closed-form EVaR for numerous continuous distributions. [31] leveraged the bilateral links between EVaR

and norms, and studied the metric induced by Banach spaces. [32] proved that the EVaR can be a finitely-valued dynamic risk measure if the moment generating function exists. In addition, several important investigations in the field can be found in [33], [34], and [35].

Substantial efforts are made to generalize the EVaR. For example, [26] introduces the g-entropic risk measures, which include the ES and the EVaR, and proves that the EVaR is computationally more efficient than the ES. [36] extend and generalize the EVaR towards the Rényi entropies. Furthermore, they replaced the relative entropy in the dual representation with different divergences and so created the f-EVaR. Also, [37] challenged the capabilities of the Adjusted Rényi EVaR to deal with different parts of the tail distribution. [38] extends the EVaR into a new coherent uncertain entropic risk measure and verifies the properties of EVaR under uncertain measure (positive homogeneity, translation invariance, monotonicity, and subadditivity under independence). The EVaR served as the basis for the introduction of other risk measures. [39] coined the mean-EVaR model that deals with uncertain random returns. [40] apply the mean EVaR results in a distributionally robust optimization model. [41] develops the multivariate EVaR as a coherent entropic risk measure that encapsulates the integrated risk of various assets in a portfolio. [42] explored the EVaR based on the Tsallis relative entropy and derived an explicit formula for the dual Tsallis VaR norm. [43] applied the Tsallis EVaR to identify the reversion measure when modeling and identifying time series with a long memory.

As we mentioned above, the main purpose of the paper is to measure the financial risks through the EVaR. Our approach is based on the assumption that the studied object (asset, stocks, commodity, index, etc.) is driven by some stochastic process. We faced the tough choice of selecting which models would be the subject of our study. We solved the dilemma by picking up models based on their capability to match the stylized facts such as sudden jumps, strong asymmetry in log-returns' distributions, volatility clustering, leverage effect, long-range dependence, etc. – see [44]. The Gaussian assumptions that lay the ground of the [45] model do not meet the stylized facts but serve as a solid ground to compare the rest of the models. We would like to see asset price dynamics different from the log-normal one and this is why we leveraged the exponential tempered stable model which is closer to the market realities since the asset price dynamics is driven by a process of infinitely small-size jumps – see the survey of [46] as well as the fundamental works of [47] and [48]. It is a stylized fact that the asset returns volatility is a stochastic process, and hence the [49] model was added to the set of models we dealt with. We would like our study to capture the combination of stochastic volatility and asset returns jumps, and this is how we enlarged the set with the model of [50] that uses compound Poisson jumps. Last but not least, we selected a stochastic volatility/tempered stable model since this model replaces the finite-number Bates jumps with jumps of infinite activity – see [51].

The approach we follow has several outstanding advantages over working directly with the empirical observations. First, using an appropriate model, we may introduce the well-observed in the markets' inherent features into the asset model. Second, the financial risks influence mainly the distributions' tails, but these observations are rare or even extremely rare. This does not allow a proper derivation of the risk measures. This is at most true when we work with a moving window of relatively small size. Third, we may use the extracted stochastic process that drives the corresponding model for other financial purposes, such as option pricing, portfolio optimization, etc. Last but not least, the stochastic process introduces a proper structure and fills the gap of limited observations.

The novelties of the paper are in several directions. First, we introduce the EVaR from a financial point of view and discuss its coherence. Furthermore, we generalize the concept of EVaR for a distribution whose moment-generating function is not well-defined everywhere. The necessary

theoretical results are proven. Second, we elaborate on the EVaR for the above-mentioned five stochastic models. We derive the domains of the moment generating functions of the log-returns for all these models and derive closed-form formulas for the EVaR. And third, we estimate the log-returns of these five models to the S&P500 index from January 2, 2003, to February 19, 2025. The approach we use is based on a moving window with a length of 1 000 days. This way, we may derive the risk structure during the whole period. Because theoretical probability density functions are not available for the above models (except for the Black-Scholes one), we approximate them by a Fourier method (Fast Fourier Transforms, FFT) through the characteristic functions. Last but not least, we analyze the empirical results for the models and compare the EVaR performance with another sophisticated market risk measure, namely the expectile risk measure (ERM) – see [52]. It is worth noting that the entropic VaR is a measure that almost immediately indicates possible market turbulence – this is strongly confirmed by the empirical results.

We chose the S&P500 index for our empirical investigation mainly since it is the most representative index in the US financial markets. In fact, it consists of the most important companies, taking into account their trading weights. Although the index is not directly tradable on the markets, many ETFs follow its structure. Furthermore, having in mind that all of the stochastic models used in this article have been widely used both in the financial theory and industry practically for all important asset types and all over the globe, we can expect with a great deal of certainty that the suggested approach can be applied beyond the S&P500 index. Some interesting evidence from Türkiye can be found in [53] and [54].

The paper is organized as follows. The EVaR risk measure is adapted to the financial purposes in Section 2. We derive the EVaR for the above-mentioned model and for a financial time series in Section 3. The calibration methodology used and its motivation are provided in Section 4. We analyze the empirical evidence in Section 5.

2 Entropic VaR for stock returns

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space – Ω is the sample space, $\mathcal{F}$ is a σ -algebra on Ω , $\mathcal{F}_t$ is a filtration over $\mathcal{F}$, and $\mathbb{P}$ is the probability measure. In financial terms, the set Ω is a collection of all trajectories of the asset prices, $\mathcal{F}$ contains the possible events, and $\mathcal{F}_t$ models the information flow w.r.t. the time. Note that $\mathbb{P}$ measures the probability of the non-sample events of $\mathcal{F}$. Also, it is the natural probability measure, also known as the real-world. We shall specify further the probability space, having in mind the stochastic processes we use for modeling the asset prices.

EVaR for financial purposes

We shall introduce the EVaR for the financial assets. It quite differs from the traditional definition in the field of operations research provided by [26]. Let ξ be a random variable that describes the asset returns. We shall use the notations $f(\cdot)$, $F(\cdot)$, $\bar{F}(\cdot)$, $q(\cdot)$, $\Psi(\cdot)$, and $M(\cdot)$ for the probability density function (PDF), the cumulative distribution function (CDF), the complementary cumulative distribution function (CCDF), the quantile function (QF), the characteristic function (CF), and the moment generating function (MGF) of ξ . Sometimes we need to mark the dependence on ξ .

The most common risk measure used in finance – the VaR – is defined as

$$VaR(\epsilon; \xi) = -\inf\{x \in \mathbb{R} : \mathbb{P}(\xi < x) > \epsilon\} \equiv -F^{-1}(\epsilon) \equiv -q(\epsilon), \quad (1)$$

for a level $\epsilon \in (0, 1)$. Usually, ϵ has small values – most common ones are 0.05, 0.025, and 0.01. We need now the following lemma – it formalizes mathematically the fact that if $g_1(x) \leq g_2(x)$

are two monotonic functions, then their inversions change the positions, i.e. $g_1^{-1}(x) \geq g_2^{-1}(x)$.

Lemma 1 Let $g_1(\cdot)$ and $g_2(\cdot)$ be two functions on the real numbers, $g_1(\cdot)$ be non-decreasing, $g_2(\cdot)$ be strictly increasing, $g_1(x) \leq g_2(\cdot)$, and the set A be the image of the real axes through the function $g_2(\cdot)$. We define the inversion of $g_1(x)$ as $g_1^{-1}(\alpha) = \inf\{x \in \mathbb{R} : g_1(x) > \alpha\}$. If $g_1(x) > \alpha$ for all real x and some $\alpha \in A$, then $g_1^{-1}(\alpha) = -\infty$. Also, if $g_1(x) < \alpha$ for all $x \in \mathbb{R}$ and some $\alpha \in A$, then $g_1^{-1}(\alpha) = +\infty$. Under these assumptions, $g_1^{-1}(\alpha) \geq g_2^{-1}(\alpha)$ at the set A .

We shall motivate now the definition of the entropic VaR as the upper bound of the classical measure VaR. Note that VaR is the inverted CDF taken with an opposite sign due to its relation with the quantile function. We shall associate the variable a with the CDF and ϵ with its inversion, respectively with the VaR. In financial terms, ϵ is the risk level that we want to measure.

Using the Chernoff inequality, also known as the exponential Markov inequality – see [55], we restrict the CDF for a positive constant $z > 0$ by

$$\begin{aligned} F(a; \xi) &= P(\xi < a) = \mathbb{P}(e^{-z\xi} > e^{-za}) \\ &\leq e^{az} \mathbb{E}[e^{-z\xi}] = e^{az} M(-z; \xi). \end{aligned} \quad (2)$$

Let us denote by $d(x, z; \xi)$ the function

$$d(x, z; \xi) := \frac{1}{z} \ln \left(\frac{M(-z; \xi)}{x} \right). \quad (3)$$

Having in mind that the inversion w.r.t. the variable a of the right hand side of inequality (2) is $-d(\epsilon, z; \xi)$ and applying Lemma 1, we deduce

$$\text{VaR}(\epsilon; \xi) = -F^{-1}(\epsilon; \xi) \leq -\sup_{z>0} -d(\epsilon, z; \xi) = \inf_{z>0} d(\epsilon, z; \xi). \quad (4)$$

Thus we reach the definition of the EVaR in the light of finance:

Definition 1 The Entropic VaR at level $\epsilon \in (0, 1)$ of random variable ξ is defined as

$$\text{EVaR}(\epsilon; \xi) = \inf_{z>0} d(\epsilon, z; \xi) \equiv \inf_{z>0} \frac{1}{z} \ln \left(\frac{M(-z; \xi)}{\epsilon} \right). \quad (5)$$

We shall denote the value at which the minimum occurs by a_ξ .

Remark 1 Definition 1 slightly deviates from the original one provided by [26]. The main difference arises from the fact that the left tail of the returns distribution is important in finance, whereas the right one is used traditionally in the field of operations research. Thus Definition 1 measures in fact the opposite random variable in the sense of [26].

Next, we discuss the support of EVaR. Obviously, if $M(-\bar{z}; \xi) < \infty$ for every $z > 0$, then EVaR is well defined. However, this is quite a strong restriction for many practical purposes. The following lemma motivates the set we use:

Lemma 2 If $M(-\bar{z}; \xi) < \infty$ for some $\bar{z} > 0$, then $M(-z; \xi) < \infty$ for every z such that $0 \leq z \leq \bar{z}$.

Proof We refer to Theorem 7.1.1. of [56].

Based on Lemma 2, we define c_{ξ} as

$$c_{\xi} = \arg \sup_{z > 0} M(-z; \xi) < \infty. \quad (6)$$

We shall say that $\mathcal{C}_{\xi} = [0, c_{\xi})$ is the negative support of the moment generating function of ξ if $M(-c_{\xi}; \xi) = \infty$. If $M(-c_{\xi}; \xi) < \infty$, then $\mathcal{C}_{\xi} = [0, c_{\xi}]$. Thus, we enlarged the support of EVAR as these random variables for which $c_{\xi} > 0$ – we shall denote this set by $\mathcal{B}$. A similar set is also discussed in [31]. Thus we modify Definition 1 in the following way:

Definition 2 For $\xi \in \mathcal{B}$ and $\epsilon \in (0, 1)$, the EVaR is defined as

$$EVaR(\epsilon; \xi) = \inf_{0 < z \leq c_{\xi}} d(\epsilon, z; \xi). \quad (7)$$

The following corollary stands:

Corollary 1 If x is a constant, then $EVaR(\epsilon; \xi + x) = EVaR(\epsilon; \xi) - x$.

Proof Having in mind the relation $M(z; \xi + x) = e^{xz} M(z; \xi)$, we derive for the function $d(\epsilon, z; \xi)$

$$d(\epsilon, z; \xi + x) = \frac{1}{z} \ln \left(\frac{M(-z; \xi + x)}{\epsilon} \right) = d(\epsilon, z; \xi) - x. \quad (8)$$

We conclude that the minimum for the EVaR is achieved at the same point a_{ξ} and it is downward shifted by x . Note that the domains of the MGFs of the random variables ξ and $\xi + x$ coincide.

To prove that the so-defined measure is coherent, we check the conditions for a coherent risk measure (see [13]), where $\nu(\cdot)$ is an image of the random variables to the reals:

(C₁) Translation Invariance: If ξ is a random variable and $x \in \mathbb{R}$, then $\nu(\xi + x) = \nu(\xi) - x$.

(C₂) Sub-additivity: If ξ_1 and ξ_2 are random variables, then $\nu(\xi_1 + \xi_2) \leq \nu(\xi_1) + \nu(\xi_2)$.

(C₃) Monotonicity: If $\xi_1 \leq \xi_2$ are random variables, then $\nu(\xi_1) \geq \nu(\xi_2)$.

(C₄) Positive homogeneity: If ξ is a random variable and $x \geq 0$, then $\nu(x\xi) = x\nu(\xi)$.

The proof is similar to Theorem 3.1 of [26], and we omit it. Note that we need to consider intervals of the form $\mathcal{C}_{\xi}$.

We shall provide the so-called dual presentation of the entropic VaR. It exists since EVaR is a coherent risk measure – see [13] and [57]. Having in mind Remark 1 and Theorem 7 of [25] we can obtain the following result:

Proposition 1 Let the set M_{ϵ} consists of these equivalent to $\mathbb{P}$ measures $\mathbb{Q}$ for which

$$\int \frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{P} \leq -\ln \epsilon. \quad (9)$$

We have

$$EVaR(\epsilon; \xi) = - \inf_{\mathbb{Q} \in M_{\epsilon}} \mathbb{E}^{\mathbb{Q}}[\xi]. \quad (10)$$

Acceptance set

The concept of the acceptance set comes from the real markets and denotes the positions that satisfy the regulator's requirements. It is defined as:

Definition 3 The acceptance set $\mathcal{A}$ associated with a risk measure ν defined on the field of the random variables is

$$\mathcal{A}_\nu = \{\xi : \nu(\xi) \leq 0\}. \quad (11)$$

We obtain the acceptance set for the EVaR-measure in the following proposition:

Proposition 2 For a random variable ξ , let $\epsilon^*(\xi)$ be defined as

$$\epsilon^*(\xi) = \inf_{0 \leq z \leq c_\xi} M(-z; \xi). \quad (12)$$

The acceptance set for the EVaR at the level ϵ is $\mathcal{A}_\epsilon = \{\xi : \epsilon^*(\xi) \leq \epsilon\}$.

Proof First, suppose that ξ is such that $\epsilon^*(\xi) \leq \epsilon$ and let $z^*(\xi)$ be

$$z^*(\xi) = \arg \inf_{0 \leq z \leq c_\xi} M(-z; \xi). \quad (13)$$

Therefore Definition 2 leads to

$$\begin{aligned} EVaR(\epsilon; \xi) &= \inf_{0 < z \leq c_\xi} \frac{1}{z} \ln \left(\frac{M(-z; \xi)}{\epsilon} \right) \\ &\leq \frac{1}{z^*(\xi)} \ln \left(\frac{M(-z^*(\xi); \xi)}{\epsilon} \right) \\ &= \frac{1}{z^*(\xi)} \ln \left(\frac{\epsilon^*(\xi)}{\epsilon} \right) \leq 0. \end{aligned} \quad (14)$$

Hence, $\xi \in \mathcal{A}_\epsilon$. Inverse, suppose that $\xi \in \mathcal{A}_\epsilon$. Therefore, $M(-z; \xi) \leq \epsilon$ for some $0 < z \leq c_\xi$. Hence, $\epsilon^*(\xi) \leq \epsilon$ too.

3 Deriving Entropic VaR

We need the following statements before continuing with deriving the EVaR.

Lemma 3 If the function $g(z)$ is monotone on a positive domain, then the function $\bar{g}(z) := g\left(\frac{1}{z}\right)$ is inverse monotone on the related set.

Proposition 3 For fixed ϵ and ξ , the function $g(z) := d(\epsilon, -z; \xi)$ has no more than one local minimum in the interval C_ξ .

Proof Using similar arguments as in Lemma 3.1 of [26] we can prove that the function $\bar{g}(z) = g\left(\frac{1}{z}\right)$ is convex and continuous on the set $\left(\frac{1}{c_\xi}, \infty\right)$. Hence, there exists a constant c , $\frac{1}{c_\xi} \leq c \leq \infty$, such that $\bar{g}(z)$ is monotone decreasing in the interval $\left(\frac{1}{c_\xi}, c\right)$ and monotone increasing in (c, ∞) . Note that the cases $c = \frac{1}{c_\xi}$ and $c = \infty$ lead to a monotone function (increasing or decreasing,

respectively) in the whole interval $\left(\frac{1}{c_{\xi}}, \infty\right)$. Using **Lemma 3** we conclude that the function $g(z)$ decreases in the interval $\left(0, \frac{1}{c_{\xi}}\right)$ and increases for $z \in \left(\frac{1}{c_{\xi}}, c_{\xi}\right)$. This finishes the proof. We are ready to prove the main result for deriving the EVaR.

Theorem 1 Let the functions $\phi(\cdot)$ and $h(\cdot)$ be defined through the MGF as

$$\begin{aligned}\phi(z; \xi) &= \ln M(-z; \xi), \\ h(z, \epsilon; \xi) &= \phi'(z)z - \phi(z) + \ln \epsilon.\end{aligned}\quad (15)$$

If $h(c_{\xi}, \epsilon; \xi) \leq 0$, then

$$EVaR(\epsilon; \xi) = \frac{1}{c_{\xi}} d(-c_{\xi}, \epsilon; \xi) \equiv \frac{1}{c_{\xi}} (\phi(c_{\xi}; \xi) - \ln \epsilon). \quad (16)$$

Otherwise, if $h(c_{\xi}, \epsilon; \xi) > 0$, then the equation $h(z, \epsilon; \xi) = 0$ has a unique solution in the interval $(0, c_{\xi})$ which is a_{ξ} and thus

$$EVaR(\epsilon; \xi) = d(-a_{\xi}, \epsilon; \xi) \equiv \frac{1}{a_{\xi}} (\phi(a_{\xi}; \xi) - \ln \epsilon). \quad (17)$$

Proof First, we observe that $d(\epsilon, 0; \xi) = +\infty$ because $M(0; \xi) = 1$ and $\epsilon < 1$. Applying **Proposition 3** we deduce that the function $d(\epsilon, -z; \xi)$ is either decreasing w.r.t. the variable z in the interval $(0, c_{\xi})$ or first decreases and then increases, having a global minimum. Using that

$$\begin{aligned}d(\epsilon, -z; \xi) &= \frac{1}{z} (\phi(z; \xi) - \ln \epsilon), \\ \frac{\partial}{\partial z} (d(\epsilon, -z; \xi)) &= \frac{h(z, \epsilon; \xi)}{z^2},\end{aligned}\quad (18)$$

we conclude that the function $h(z, \epsilon; \xi)$ is always negative for $z \in (0, c_{\xi})$ or it has a unique root in this interval, $h(z, \epsilon; \xi) < 0$ before it, and $h(z, \epsilon; \xi) > 0$ after it. If $h(c_{\xi}, \epsilon; \xi) \leq 0$, then $h(z, \epsilon; \xi) < 0$ in the whole interval $(0, c_{\xi})$ and thus $d(\epsilon, -z; \xi)$ decreases w.r.t. z . Alternatively, if $h(c_{\xi}, \epsilon; \xi) > 0$, then equation $h(z, \epsilon; \xi) = 0$ has a unique solution in $(0, c_{\xi})$ and it is the minimizer of $d(\epsilon, -z; \xi)$. We finish the proof having in mind **Definition 2**.

Empirical EVaR

Suppose that we have N observations for the log-returns, $r_i, i = 1, 2, \dots, N$. We need to obtain the MGF function of the empirical distribution. The empirical MGF can be obtained by averaging:

$$M(z) = \frac{\sum_{n=1}^N e^{zr_n}}{N}. \quad (19)$$

Alternatively, we can use the CDF that, for the empirical observations, can be derived as the number of observations lower than z divided by N

$$F(z) = \frac{\#\{r_n < z\}}{N}. \quad (20)$$

Denoting by r_{min} and r_{max} the lowest and highest observations, the equalities $F(r_{min}) = 0$, $F(r_{max}) = 1$, and integrating by parts, we obtain

$$M(z) = \int_{r_{min}}^{r_{max}} e^{uz} dF(u) = e^{zr_{max}} - z \int_{r_{min}}^{r_{max}} e^{uz} F(u) du. \quad (21)$$

The models

We shall index all studied terms by the used models instead of the random variables – Black-Scholes (BS), Exponential Tempered Stable (ETS), Heston, Stochastic Volatility with Compound Poisson Jumps (Bates, SV/CP), and Stochastic Volatility with Tempered Stable Jumps (SV/TS).

Black-Scholes model

The asset price values are presented by the log-normal process

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad (22)$$

and thus the asset t -returns

$$Y_t := \ln \left(\frac{S_t}{S_0} \right) = \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t, \quad (23)$$

are normally distributed with parameters $\left\{ \left(\mu - \frac{\sigma^2}{2} \right) t, \sigma \sqrt{t} \right\}$. The Gaussian moment generating function is always well-defined, and it is

$$M(z) = e^{t \left(\left(\mu - \frac{\sigma^2}{2} \right) z + \frac{\sigma^2 z^2}{2} \right)}. \quad (24)$$

Hence, $c_{BS} = +\infty$. Thus functions (15) turn into

$$\begin{aligned} \phi(z; \xi) &= t \left(- \left(\mu - \frac{\sigma^2}{2} \right) z + \frac{\sigma^2 z^2}{2} \right), \\ h(z, \epsilon; \xi) &= \phi'(z) z - \phi(z) + \ln \epsilon. \end{aligned} \quad (25)$$

We conclude that the values of minimizer a_{BS} and the $EVaR$ are

$$\begin{aligned} a_{BS} &= \frac{1}{\sigma} \sqrt{-\frac{2 \ln \epsilon}{t}}, \\ EVaR(\epsilon; BS) &= \left(\frac{\sigma^2}{2} - \mu \right) t + \sigma \sqrt{-2t \ln \epsilon}. \end{aligned} \quad (26)$$

Exponential tempered stable model

In this model, the log-returns follow a tempered stable process, i.e., a Lévy process with drift μ and Lévy measure

$$\nu(u) = \frac{c_1}{(-u)^{1+\alpha_1}} e^{\lambda_1 u} I_{u < 0} + \frac{c_2}{u^{1+\alpha_2}} e^{-\lambda_2 u} I_{u > 0}, \quad (27)$$

where $c_{1,2}$, $\lambda_{1,2}$ are positive reals and $\alpha_{1,2} \in (0, 2) - c_1, \lambda_1$, and α_1 describe the downward jumps whereas c_2, λ_2 , and α_2 are for the upward ones. The moment generating function is well defined in the interval $(-\lambda_1, \lambda_2)$ and thus $c_{ETS} = \lambda_1$. It can be written as

$$M_{TS}(z) = e^{t(\mu z + \psi_1(z) + \psi_2(z))}, \quad (28)$$

where

$$\begin{aligned} \psi_1(z) &= \begin{cases} \Gamma(-\alpha_1) \lambda_1^{\alpha_1} c_1 \left(\left(1 + \frac{z}{\lambda_1}\right)^{\alpha_1} - 1 - \frac{z\alpha_1}{\lambda_1} \right), & \text{if } \alpha_1 \neq 1, \\ -zc_1 + c_1(\lambda_1 + z) \ln \left(1 + \frac{z}{\lambda_1}\right), & \text{if } \alpha_1 = 1, \end{cases} \\ \psi_2(z) &= \begin{cases} \Gamma(-\alpha_2) \lambda_2^{\alpha_2} c_2 \left(\left(1 - \frac{z}{\lambda_2}\right)^{\alpha_2} - 1 + \frac{z\alpha_2}{\lambda_2} \right), & \text{if } \alpha_2 \neq 1, \\ zc_2 + c_2(\lambda_2 - z) \ln \left(1 - \frac{z}{\lambda_2}\right), & \text{if } \alpha_2 = 1. \end{cases} \end{aligned} \quad (29)$$

Hence, the function $\phi(\cdot; ETS)$ from (15) is $\phi(z; ETS) = t(-\mu z + \psi_1(-z) + \psi_2(-z))$. Therefore, $\phi'(z; ETS) = -t(\mu + \psi_1'(-z) + \psi_2'(-z))$ where

$$\begin{aligned} \psi_1'(z) &= \begin{cases} \Gamma(-\alpha_1) \lambda_1^{\alpha_1-1} c_1 \alpha_1 \left(\left(1 + \frac{z}{\lambda_1}\right)^{\alpha_1-1} - 1 \right), & \text{if } \alpha_1 \neq 1, \\ c_1 \ln \left(1 + \frac{z}{\lambda_1}\right), & \text{if } \alpha_1 = 1, \end{cases} \\ \psi_2'(z) &= \begin{cases} \Gamma(-\alpha_2) \lambda_2^{\alpha_2-1} c_2 \alpha_2 \left(\left(1 - \frac{z}{\lambda_2}\right)^{\alpha_2-1} + 1 \right), & \text{if } \alpha_2 \neq 1, \\ -c_2 \ln \left(1 - \frac{z}{\lambda_2}\right), & \text{if } \alpha_2 = 1. \end{cases} \end{aligned} \quad (30)$$

Several cases can be distinguished – we prove the results by the proposition below.

Proposition 4 Let $D(x)$ be the digamma function and $D^{-1}(x)$ be its inversion in the interval $(-2, -1)$. Note that it exists because the digamma function increases between minus and plus infinity in this interval. Let the constant b be defined as $b = c_1 \Gamma(D^{-1}(\ln \lambda_1)) \lambda_1^{-D^{-1}(\ln \lambda_1)}$. The following statements hold:

- i. If $b - \psi_2'(-\lambda_1) \lambda_1 - \psi_2(-\lambda_1) + \frac{\ln \epsilon}{t} > 0$, then $a_{ETS} < c_{ETS}$.
- ii. If $b - \psi_2'(-\lambda_1) \lambda_1 - \psi_2(-\lambda_1) + \frac{\ln \epsilon}{t} \leq 0$, then there exist two constants b_1 and b_2 , $1 < b_1 < -d^{-1}(\ln \lambda_1) < b_2 < 2$, for which $a_{ETS} < c_{ETS}$ when $\alpha_1 \in (0, b_1) \cup (b_2, 2)$ and $a_{ETS} = c_{ETS}$ when $\alpha_1 \in [b_1, b_2]$.

The EVaR is obtained through Theorem 1.

Proof The proof is provided in Appendix A.

Heston model

Next, we discuss the stochastic volatility model introduced by [49]. It is presented by the following dynamics

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t} S_t dB_t, \\ dV_t &= \xi(\eta - V_t) dt + \theta \sqrt{V_t} d\tilde{B}_t. \end{aligned} \quad (31)$$

The variable ξ is the speed of reversion to the value η , and θ is the volatility of the volatility. In addition, the Brownian motions B_t and $\tilde{B}_t$ are correlated with the coefficient ρ . Thus the Heston

model is driven by five parameters μ , ξ , η , θ , and ρ . The negative support is not obvious for this distribution. The detailed survey can be found in [58]. We need the following result, which is formulated as Theorem 3.1, point 4, from the study [58]:

Proposition 5 Let the quadratic function $p(x)$ be defined as

$$\begin{aligned} p(x) &= (\xi - \rho\theta x)^2 + \theta^2 (x - x^2) \\ &\equiv -x^2\theta^2 (1 - \rho^2) + x\theta (\theta - 2\rho\xi) + \xi^2. \end{aligned} \quad (32)$$

It has two real roots, possibly infinitely small (large), $-\infty \leq x_1 < 0 < x_2 \leq \infty$, $p(x) > 0$ when $x \in (x_1, x_2)$, and $p(x) < 0$ if $x < x_1$ or $x > x_2$:

$$x_{1,2} = \frac{\theta - 2\rho\xi \pm \sqrt{\theta^2 - 4\rho\theta\xi + 4\xi^2}}{2\theta(1 - \rho^2)}. \quad (33)$$

Let $P(x) = \sqrt{|p(x)|}$ and the function $f(\cdot)$ be defined as

$$\begin{aligned} f(x) &= f_1(x) + (\xi - \rho\theta x) f_2(x), \\ f_1(x) &= \cosh\left(\frac{P(x)t}{2}\right) I_{x \in (x_1, x_2)} + \cos\left(\frac{P(x)t}{2}\right) I_{x \notin (x_1, x_2)}, \\ f_2(x) &= \frac{1}{P(x)} \left[\sinh\left(\frac{P(x)t}{2}\right) I_{x \in (x_1, x_2)} + \sin\left(\frac{P(x)t}{2}\right) I_{x \notin (x_1, x_2)} \right]. \end{aligned} \quad (34)$$

The MGF of the log-returns can be written as

$$M(z; v) = e^{\omega(z) + v\kappa(z)}, \quad (35)$$

where v is the initial value of the volatility and the functions $\omega(\cdot)$ and $\kappa(\cdot)$ are defined as

$$\begin{aligned} \omega(z) &= \frac{2\xi\eta}{\theta^2} \left((\xi - \rho\theta z) \frac{t}{2} - \ln f(z) \right) + t\mu z, \\ \kappa(z) &= (z^2 - z) \frac{f_2(z)}{f(z)}. \end{aligned} \quad (36)$$

Let x^* be the highest root of $f(\cdot)$ which is less than x_1 . The abscissa of EVaR is $c_{Heston} = -x^*$.¹ Functions (15) related to the MGF of the log-price in its domain can be written as

$$\begin{aligned} \phi(z; \xi) &= -x_0 z - \mu t z + \frac{2\xi\eta}{\theta^2} \left((\rho\theta z + \xi) \frac{t}{2} - \ln f(-z) \right) + v_0 \frac{z + z^2}{f(-z)} f_2(-z), \\ h(z, \epsilon; \xi) &= \phi'(z) z - \phi(z) + \ln \epsilon \\ &= \ln \epsilon + \frac{2\xi\eta}{\theta^2} \left(-\frac{\xi t}{2} + \frac{f'(-z)}{f(-z)} z + \ln f(-z) \right) \end{aligned}$$

1 In fact, x^* is the left abscissa of convergence of the MGF of the log-price in the Heston model. The right one is more variable – the complete exposition is provided in Theorem 3.1, points 1-3 of [58].

$$\begin{aligned}
& + v z^2 \left(\frac{f(-z) + (1+z)f'(-z)}{f^2(-z)} f_2(-z) - \frac{1+z}{f(-z)} f_2'(-z) \right), \\
f'(x) &= f_1'(x) - \rho \theta f_2(x) + (\xi - \rho \theta x) f_2'(x), \\
f_1'(x) &= \frac{t}{4} f_2(x) p'(x), \\
f_2'(x) &= \operatorname{sgn}(p(x)) \frac{p'(x)}{2P^2(x)} \left[\frac{t}{2} f_1(x) - f_2(x) \right], \\
p'(x) &= -2x\theta^2 (1 - \rho^2) + \theta (\theta - 2\rho\xi).
\end{aligned} \tag{37}$$

We obtain the minimizer that determines the value of EVaR (a_{ξ} or c_{ξ}) through *Theorem 1*, more precisely via the sign of $h(c_{\xi}, \epsilon; \xi)$.

Stochastic volatility jump models

Assume now that the asset exhibits jointly stochastic volatility and jumps – $S_t = S_t^c e^{Y_t}$, where S_t^c is the continuous part given by dynamics (31) and Y_t is a Lévy process independent of the Brownian motion. In the model of [50], Y_t is a compound Poisson process with intensity λ and (α, β) -normally distributed jumps. The MGF of the jump process is always well-defined, and it is $M(z) = e^{t\psi_{CP}(z)}$, where $\psi_{CP}(u)$ is

$$\psi_{CP}(z) = \lambda \left(e^{\frac{\beta^2 z^2}{2} + \alpha z} - 1 \right). \tag{38}$$

We consider, in addition, the stochastic volatility jump model suggested by [51] – the jumps are presented by a tempered stable process discussed in Section 3. We can formulate the following results:

Proposition 6 *The domain of the MGF of the log-returns of a stochastic volatility jump model is the intersection between the domains of the continuous and jump parts. Particularly, the support for the Bates model coincides with that for Heston's model. Thus, $c_{\text{Bates}} = c_{\text{Heston}}$ and $c_{\text{SVTS}} = \min\{c_{\text{Heston}}, \lambda_1\}$, where the subscript SVTS stands for the stochastic volatility model with tempered stable jumps.*

Proof The proof is based on the fact that the MGF of the product of independent random variables is the product of their MGFs.

Having in mind Proposition 6, we derive the minimizer for the EVaR as well as its own value through Theorem 1. Functions (15) turn into

$$\begin{aligned}
\phi(z; \epsilon) &= \ln M_{\text{Heston}}(-z) + \ln M_J(-z; \xi), \\
h(z, \epsilon) &= \left[\phi'_{\text{Heston}}(z) + \phi'_J(z) \right] z - \phi_{\text{Heston}}(z) - \phi_J(z) + \ln \epsilon.
\end{aligned} \tag{39}$$

The subscript J stands for the jump part. The terms for the Bates model are

$$\begin{aligned}
\phi_J(z) &= t\lambda \left(e^{-\left(\frac{\delta^2 z^2}{2} + \left(k - \frac{\delta^2}{2}\right)z\right)} - 1 \right), \\
h(z, \epsilon; \xi) &= \left[\phi'_{\text{Heston}}(z) + \phi'_J(z) \right] z - \phi_{\text{Heston}}(z) - \phi_J(z) + \ln \epsilon, \\
\phi'_J(z) &= -t\lambda e^{-\left(\frac{\delta^2 z^2}{2} + \left(k - \frac{\delta^2}{2}\right)z\right)} \left(\delta^2 z + k - \frac{\delta^2}{2} \right).
\end{aligned} \tag{40}$$

For the model of [51], we need to combine the results of Section 3 (exponential tempered stable model) and Section 3 (Heston model).

Averaging w.r.t. the volatility

Several authors propose an approach for eliminating the initial value of the volatility that appears in MGF (35) – see [59], [51], and [52]. This is necessary because the volatility is a hidden market parameter, and there is no consensus between practitioners and researchers on how to extract it. On the other hand, the Cox-Ingersoll-Ross (CIR) process admits a (α, β) -Gamma stationary distribution, where $\alpha = \frac{2\xi}{\theta^2}$ and $\beta = \frac{2\xi\eta}{\theta^2}$. Its PDF is

$$f_{\gamma}(u) = \frac{\alpha^{\beta}}{\Gamma(\beta)} u^{\beta-1} \exp(-\alpha u) I_{u \geq 0}. \quad (41)$$

The above-mentioned authors suggest the volatility to be averaged with respect to this Gamma distribution, and thus, the averaged MGF turns into

$$\begin{aligned} M_{Heston}^{av}(z) &= \int_0^{\infty} M_{Heston}(z; v) f_{\gamma}(v) dv \\ &= \int_0^{\infty} e^{\omega(z) + \kappa(z)v} f_{\gamma}(v) dv = e^{\omega(z)} M_{\gamma}(\kappa(z)), \end{aligned} \quad (42)$$

where $M_{\gamma}(\cdot)$ is the MGF of the Gamma distribution

$$M_{\gamma}(u) = \left(1 - \frac{u}{\alpha}\right)^{-\beta} \equiv \left(1 - \frac{\theta^2 u}{2\xi}\right)^{-\frac{2\xi\eta}{\theta^2}}. \quad (43)$$

Proposition 3.1 from [60] gives the existence of a random variable whose MGF is given by formula (42) – we shall call it *averaged log-returns*. Following Theorem 3.1 from [60], we derive the left abscissa of convergence c_{av} of the MGF:

Proposition 7 For a fixed t , the equation $\kappa(t, x) = \frac{2\xi}{\theta^2}$ has a unique root in the interval $(-c_{Heston}, 0)$. The boundary c_{av} is the opposite value of this root.

The function $h(z, \epsilon; \xi)$ that describes the minimizer of the EVaR turns into

$$\begin{aligned} h(z, \epsilon; \xi) &= \phi'(z)z - \phi(z) + \ln \epsilon, \\ \phi(z, \epsilon; \xi) &= \omega(-z) - \frac{2\xi\eta}{\theta^2} \ln \left(1 - \frac{\theta^2 \kappa(-z)}{2\xi}\right), \\ \phi'(z, \epsilon; \xi) &= -\omega'(-z) - \frac{2\xi\eta}{2\xi - \theta^2 \kappa(-z)} \kappa'(-z), \\ \omega'(x) &= -\frac{2\xi\eta}{\theta^2} \left(\frac{\rho\theta t}{2} + \frac{f'(x)}{f(x)}\right) + t\mu, \\ \kappa'(x) &= \frac{(2x-1)f_2(x) + (x^2-x)f'_2(x)}{f(x)} - (x^2-x) \frac{f_2(x)f'(x)}{f^2(x)}. \end{aligned} \quad (44)$$

The derivatives $f'(x)$ and $f'_2(x)$ can be found in formulas (37).

4 Models calibration

In Table 1, we provide the *EVaR*'s based on the confidence level $\epsilon \in \{0.001, 0.01, 0.025, 0.05, 0.75, 0.99\}$ and the parameters reported in Table 2 of [52]. We can observe in Figure 1a that there are several similarities between the model *EVaR*s and the empirical ones, but the match is not perfect. To have a more precise conclusion, we calculate the *EVaR*s produced by the models calibrated in [52]. The calibration is prepared through the daily observations of the S&P 500 index for the period between September 25, 2008, and July 01, 2022, and 1000-sized moving window. The results are presented in Figure 1. We can see that although there are some similarities, the discrepancies are substantial. Several reasons may lead to this inaccuracy – the most likely one is the existence of over-fitting. To check this hypothesis, we recalibrate the models, defining a cost function consisting of two parts – one for the distribution's PDF and another for the *EVaR*. Also, we update the historical sample considering the period between January 2, 2003, and February 19, 2025. We use the classical procedure for extracting the empirical PDF. The interval $(-0.23, 0.23)$ for the log-returns is divided into 230 bins with length 0.002. The interval is chosen in that way because the lowest log-return ever is -0.2290 , observed at October 19, 1987 – the so-called Black-Monday. The empirical PDF values at the centers of the sub-intervals are obtained by counting the number of observations that fall in them. The PDF values for the log-returns in the theoretical models, except the Black-Scholes one, are obtained by the Fast Fourier Transform because only the characteristic functions are available. We present them in Appendix B. As we mentioned in Section 3, the initial volatility for the stochastic volatility models is a hidden market parameter – we suggest removing it from the calibration methodology by averaging w.r.t. its stationary distribution. We use the standard least squares error method to compare the theoretical and empirical PDFs. The second term of the cost function is related to the deviations from the empirical *EVaR*s produced by the theoretical models. For a relative equalization of the weights of both components, we multiply the *EVaR*-part by 10^3 . Thus, we reach the minimization problem

$$\min_{\gamma \in Y} \left\{ \sum_{n=1}^{230} (p_{th,n} - p_{emp,n})^2 + 10^3 \sqrt{\sum_{n=1}^6 (EVaR_{th}(\epsilon_n) - EVaR_{emp}(\epsilon_n))^2} \right\}, \quad (45)$$

where Y is the set of all possible parameters for the theoretical model, $\gamma \in Y$, the symbol p is used for the PDFs, and by a subscript, the empirical and theoretical terms.

We derive the model's parameters without applying either penalization or regularization since we are not challenging the stability of the model performance over such a large period. On the contrary, we would like to see how volatile the parameters are.

Table 1. *EVaR*s based on Table 2 of [52]

Parameter	Empirical	BS	ETS	Heston	Bates	SV/TS
0.001	0.1144	0.0388	0.1139	0.1527	0.1061	0.1214
0.01	0.0857	0.0317	0.0795	0.1045	0.0762	0.0847
0.025	0.0726	0.0284	0.0657	0.0852	0.0638	0.0699
0.05	0.0212	0.0124	0.0179	0.0208	0.0184	0.0188
0.75	0.0116	0.0080	0.0100	0.0111	0.0103	0.0104
0.99	0.0015	0.0016	0.0013	0.0015	0.0014	0.0014

5 Analyzing the numerical results

Calibration results

The calibrated parameters for the five models based on the whole data are reported in Table 4. The generated PDFs can be compared to the empirical values in Figure 1b. The values of the cost function (45) as well as its two components can be seen in the first part of Table 2. We conclude that all models except the Black-Scholes one capture the shape of the empirical PDF. Some unexpected results, such as the best ETS fit, can be explained by the challenges that stand in the multidimensional optimization. The value function during the period, as well as its two components, can be seen in Figure 2a, Figure 2b, and Figure 2c.

Table 2. Deviations, Expectile Risk Measures, Expected Shortfall, VaR

Deviations	Empirical	BS	ETS	Heston	Bates	SV/TS
total	—	108.5314	14.4035	16.7178	19.0651	15.1364
PDF	—	92.9494	8.0434	9.6454	14.0027	7.5923
EVaR	—	15.5820	6.3601	7.0724	5.0624	7.5441
EVaR	Empirical	BS	ETS	Heston	Bates	SV/TS
0.001	0.1121	0.0994	0.1164	0.1168	0.1158	0.1173
0.01	0.0835	0.0810	0.0821	0.0820	0.0827	0.0821
0.025	0.0706	0.0724	0.0682	0.0680	0.0690	0.0679
0.05	0.0602	0.0652	0.0576	0.0572	0.0583	0.0571
0.75	0.0539	0.0606	0.0512	0.0508	0.0519	0.0506
0.99	0.0014	0.0031	0.0015	0.0012	0.0021	0.0015
ERM	Empirical	BS	ETS	Heston	Bates	SV/TS
0.001	0.0564	0.0649	0.0507	0.0496	0.0510	0.0489
0.01	0.0271	0.0456	0.0288	0.0277	0.0252	0.0276
0.025	0.0192	0.0370	0.0211	0.0202	0.0183	0.0202
0.05	0.0141	0.0300	0.0157	0.0149	0.0139	0.0151
ES	Empirical	BS	ETS	Heston	Bates	SV/TS
0.001	0.1055	0.0889	0.0785	0.0773	0.0841	0.0760
0.01	0.0518	0.0711	0.0515	0.0502	0.0467	0.0495
0.025	0.0380	0.0624	0.0404	0.0392	0.0348	0.0388
0.05	0.0293	0.0550	0.0322	0.0310	0.0275	0.0309
VaR	Empirical	BS	ETS	Heston	Bates	SV/TS
0.001	0.0792	0.0828	0.0683	0.0670	0.0706	0.0659
0.01	0.0347	0.0622	0.0395	0.0381	0.0322	0.0378
0.025	0.0248	0.0523	0.0285	0.0273	0.0234	0.0273
0.05	0.0174	0.0438	0.0205	0.0195	0.0177	0.0197

The reported values in the second part of Table 2 for the entropic VaR confirm that all non-lognormal models capture the risk profile of the S&P 500 index. Furthermore, we present the other risk measures discussed in this paper – VaR, ES, and ERM – in Table 2. We can see that despite some deviations, the calibrated models generate values for these measures that are relatively close to the empirical ones. The EVaRs of the five models we deal with can be seen in Figure 1c. We use a moving window again with 1000 positions, and thus the period consists of 4569 observations between December 21, 2006, and February 19, 2025. First, it is worth underlining that the profile of the empirical EVaR very clearly differentiates the zones with a heightened level of market

risk. Furthermore, all models except the Black-Scholes model more or less follow this behavior. This shows that the entropic VaR is a risk measure that almost immediately indicates market turbulence.

We consider further the other discussed risk measures, ERM, ES, and VaR, calculating them based on the already calibrated parameters via the moving window and criteria (45). The results for the expectile risk measure can be viewed in Figure 1d, for the expected shortfall in Figure 1e, and for the VaR in Figure 1f. We can conclude that despite some deviations, the calibrated models follows the behavior of the ERM, ES, and VaR in addition to the entropic VaR. Hence, the calibration method we suggest captures not only the distributions and EVars, but and some other inherent properties coming from the expectile risk measure, expected shortfall, and VaR. All these results confirm that the approach can be used for avoiding over-fitting as well as for capturing the intrinsic properties of the financial data.

Another important question that stands is how stable are the estimations when we change the equalization coefficient in cost function (45). We check this considering the weight coefficient to be equal to $c10^3$ for $c = 0.9$ and $c = 1.1$. We re-calibrate only the Heston's stochastic volatility model since it is the main one in our investigation. The results are provided in Table 3. We can conclude that the new estimations are relatively close to the original ones.

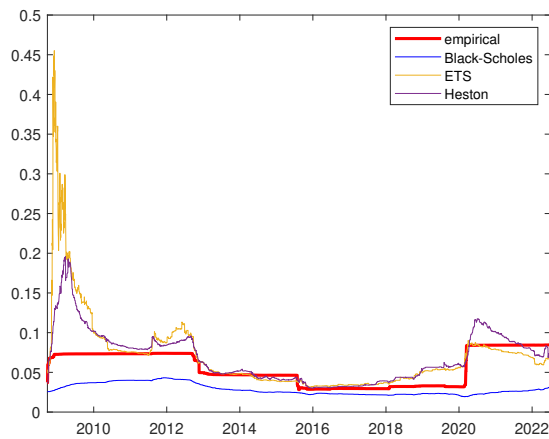
Table 3. Stability results

Parameter	μ	ζ	η	θ	ρ
$c = 0.9$	0.1953	148.6696	0.0416	5.7540	-0.0044
$c = 1$	0.1946	142.7981	0.0419	5.6212	-0.0041
$c = 1.1$	0.1935	135.1603	0.0422	5.4456	-0.0036

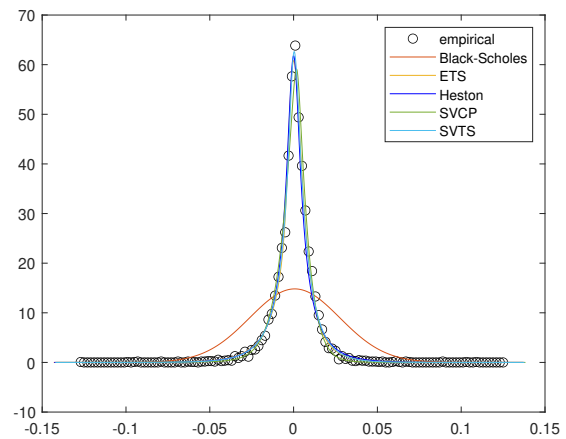
Model behavior

We choose to comment on the parameters inferred for the Heston model because it is based on the stochastic volatility, which we observe on the financial markets, while the Black-Scholes model is based on the assumption of a constant volatility. Also, the model is well-known among the academic circles and the practitioners, and this secures a solid ground to compare our results with previous studies and a large enough target group for the article. The Heston model paves the way to understanding the more sophisticated models elaborated in the article.

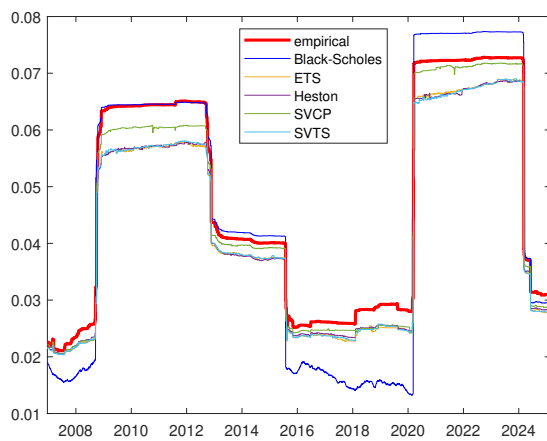
A general observation is that the Global Financial Crisis 2007 - 2009 (GFC) and COVID-19 left footprints on the profile of every parameter. Figure 3 confirms that Heston parameters for the EVaR reacted to the GFC and to the COVID-19 pandemic. The most important observation regarding μ is that it changes its sign; it reaches the lowest values during the COVID-19 pandemic and has record high values from the end of 2012 to the end of 2016. The speed of the mean reversion ζ is a very volatile parameter with a minimum value of 0.3279 at the very last day of the considered period and a maximum value of 910.3188 observable from March 20, 2020, to April 21, 2020. The parameter η is of very low value until the break of the GFC; also, from August 2015 till the break of the COVID-19; its value decreases from the end of 2012 to mid-2015 and increases steadily during the remaining periods. We observe that the profiles of ζ and θ are surprisingly identical, despite different scales - see Figure 2e. The correlation coefficient ρ has a very stable behavior through the calm market periods and is significantly volatile in the turbulence. Let us also consider the impact of Heston parameters evolution on the distribution of the log-returns in light of [61]. We know that the log-returns are positively skewed when ρ is positive and negatively skewed when ρ is



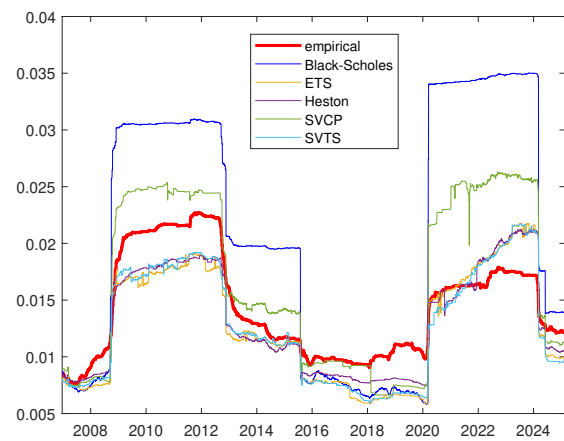
(a) EVaRs for data from [52]



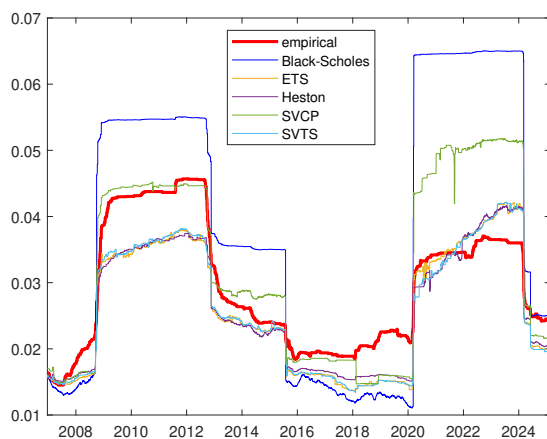
(b) Calibrated PDFs



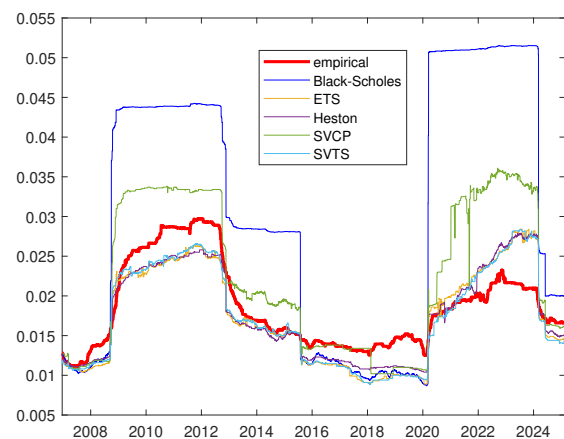
(c) Calibrated EVaRs



(d) ERM



(e) ES

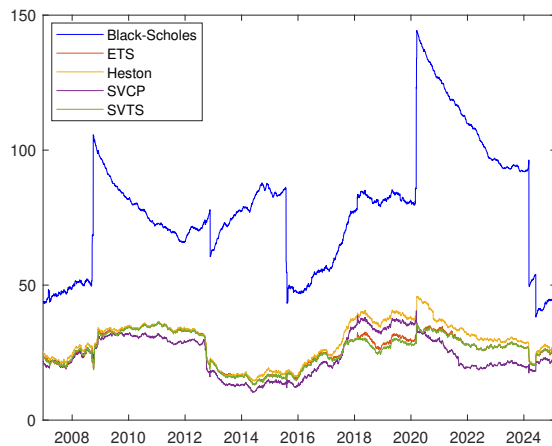


(f) VaRs

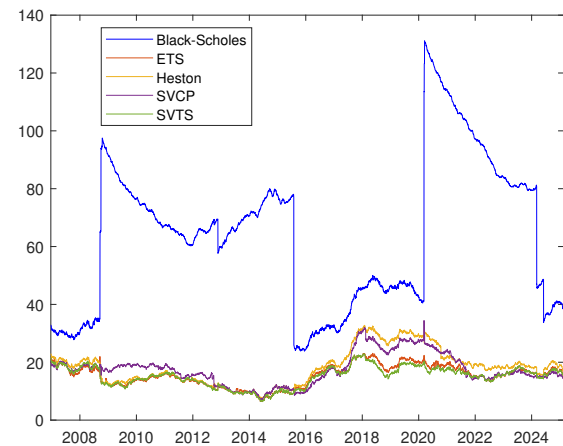
Figure 1. The calibrated PDFs and the risk measures

negative. Hence, we anticipate that the related distribution is negatively skewed except in the period after March 2024. We know that a high θ results in high distribution kurtosis. Hence, we anticipate fat tails for the whole period except the first quarter after the beginning of the COVID-19 pandemic.

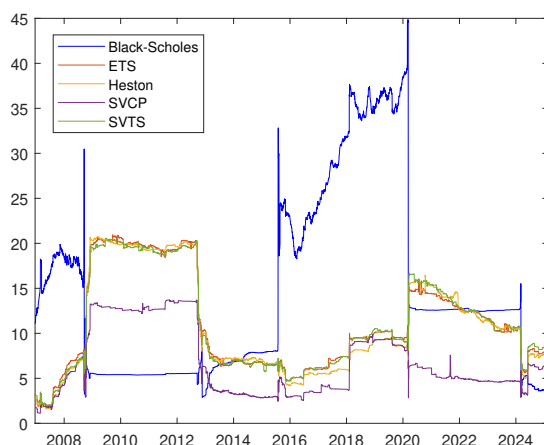
Last but not least, we compare the market risk measured by the expectile risk measure and the entropic VaR through the results reported in see [52]. We take into account that the empirical data for this investigation consists of the daily observations of the S&P500 index between Sept 25, 2008 to July 1, 2022 – totally of 3466 log-returns. We use this data to equalize the historical samples. First, we can observe that the entropic VaR better reacts to exogenous events like the GFC and the COVID19 – see Figure 2d.



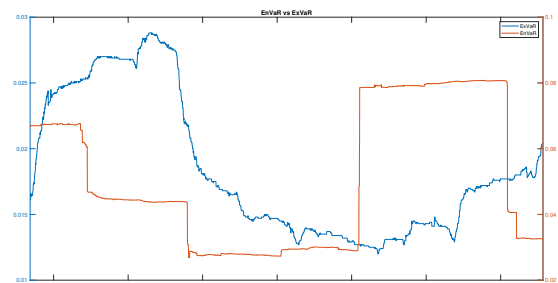
(a) Value function



(b) Value function – PDFs



(c) Value function – EVaRs



(d) ERM vs EVaR

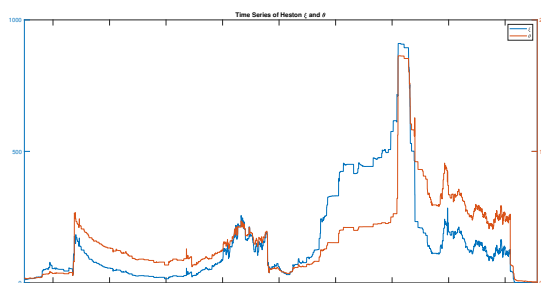
(e) Heston's ζ vs θ

Figure 2. The value function and some comparisons

On the other hand, Figure 4 presents the joint plot of the parameters. We see a good match for the time drift μ . Also, we note that ξ and θ are of very different scales. The variance volatility θ is the same from the second half of 2015 to the beginning of COVID-19. During the rest of the period, we see opposite values. Furthermore, we see that the correlation coefficient for the entropic VaR is rather stable, whereas it is more volatile for the expectile risk measure.

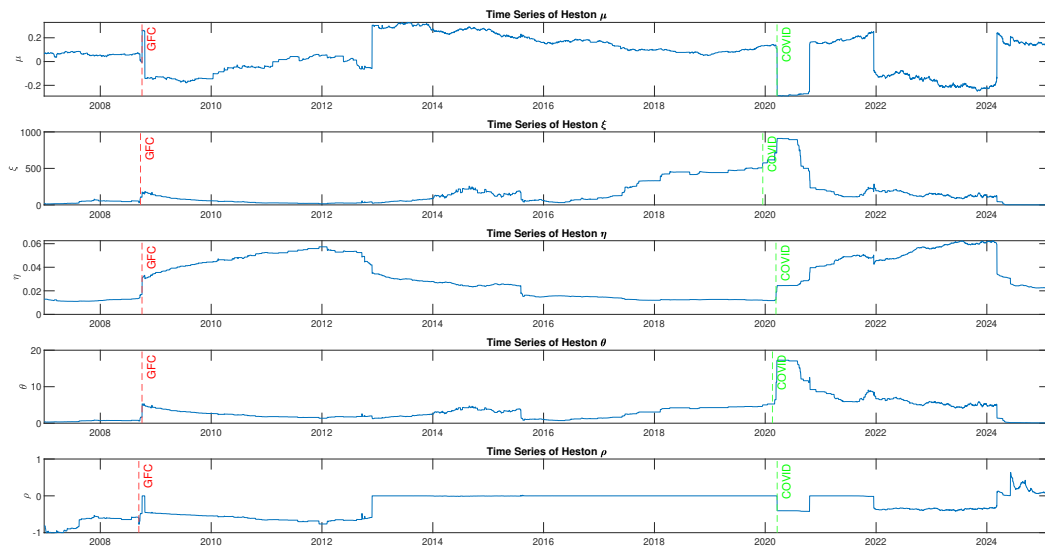


Figure 3. EVaR Heston parameters

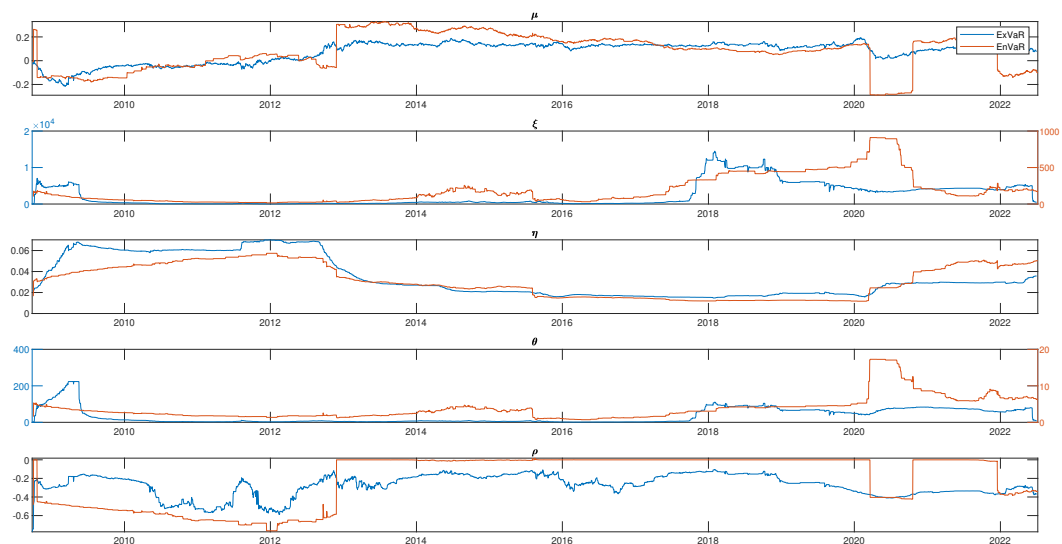


Figure 4. Comparing Heston parameters for ExVaR and EVaR

Comparison between the risk measures

We now discuss some of the main differences between traditional quantile-based risk measures (VaR and AVaR) on the one hand and Entropic VaR on the other. The VaR is the most commonly used risk measure. Its main advantages are its intuitiveness and computational ease – in fact, it is the quantile. From the theoretical point of view, the main shortcoming is that VaR is not coherent – it is not subadditive. The economic meaning of $VaR(\epsilon)$ is the maximum loss that one can expect with $100\epsilon\%$ of confidence. The main economic shortcoming is that it does not say anything about what happens beyond the threshold ϵ . On the other hand, the ES-measure is defined as the average possible losses beyond the level – note that its alternative name is Averaged-Value-at-Risk (AVaR). This indicates that $VaR < ES$. On the other hand, the way we define the Entropic-VaR makes it the most conservative amongst the considered risk measures. This leads to a strong insurance against the rare but catastrophic events. This leads to the order $VaR < ES < EVaR$. We illustrate all these in Figure 5. We chose the level $\epsilon = 0.05$ for the already calibrated models of Black-Scholes and Heston – their parameters are reported in Table 4. First, remind that the Black-Scholes model produces Gaussian log-returns – see Figure 5a. The VaR is presented by the red line. The shaded part has a surface just 0.05 – it consists of all risks with probabilities lower than 0.05. The ES and EVaR are given by the green and cyan lines. The above-mentioned order is confirmed – note that the measure values have to be taken with the opposite sign. In Figure 5b, we illustrate the results for Heston's stochastic volatility model. We can observe that the deviations between the discussed three measures are significantly larger for this model. This happens because Heston's model captures significantly better the heavy tails in the empirical data. On the other hand, the expectile risk measure is defined in a significantly different way, and direct comparisons would not be correct. Its main advantage is that it takes into account both the right and left tails. From economical point of view, this means both possible profits and losses influence the ERM. This holds for the EVaR too, but to a much lesser extent, since the minimization in Definition 2 is in the negative domain of the MGF.

Table 4. Parameters for Black-Scholes, exponential tempered stable, Heston, Bates, and SV/TS models

Parameter	BS	ETS	Heston	Bates	SV/TS
μ	0.2632	0.0767	0.1946	-0.0829	0.0616
ξ	-	-	142.7981	28.1853	37.4461
η	-	-	0.0419	0.0225	4.8209E-04
θ	-	-	5.6212	0.9738	0.0161
ρ	-	-	-0.0041	-0.7944	-0.2295
α_1	-	6.1266E-06	-	-	0.2402
α_2	-	1.0187	-	-	0.407
c_1	-	112.4077	-	-	37.3946
c_2	-	0.6192	-	-	18.7313
λ_1	-	71.5889	-	-	67.5161
λ_2	-	34.2866	-	-	79.1169
σ	0.4274	-	-	-	-
λ	-	-	-	3.5253	-
k	-	-	-	-0.0154	-
δ	-	-	-	0.0354	-

Last but not least, let us discuss the observation that the EVaR reacts significantly faster to market turbulence. This measure is highly sensitive to extremely rare downward events since it is based

on the exponential moments through the distribution's MGF. This makes the EVaR the most conservative risk measure that preserves against large risks.

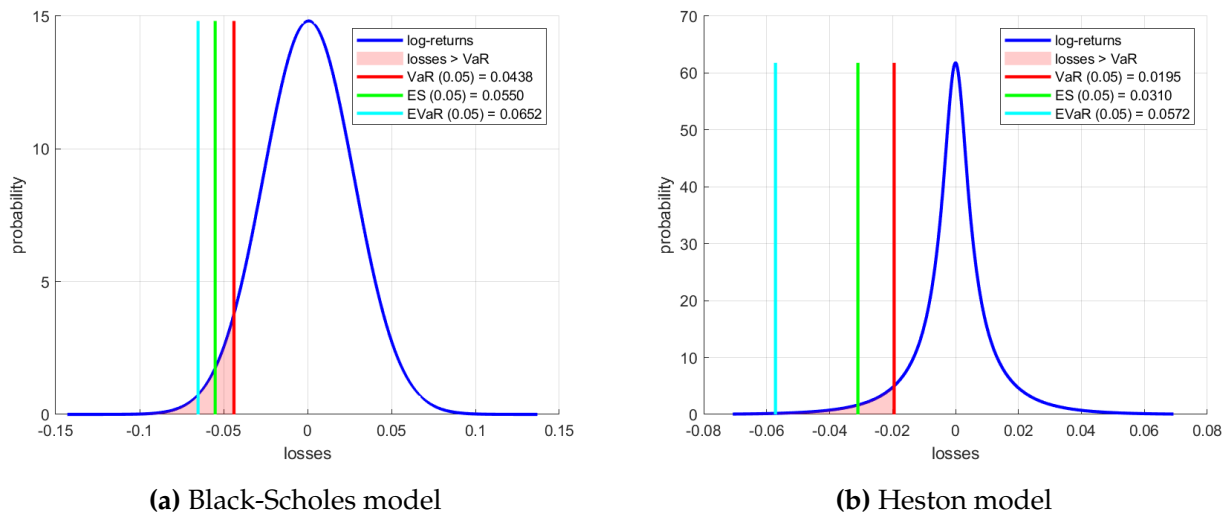


Figure 5. VaR vs. ES vs. EVaR

6 Conclusions

The paper investigates the entropic VaR as a coherent risk measure in the finance area. It is compared with the two main measures that are widely used both in financial theory and practice – Value-at-Risk (VaR) and Expected Shortfall (ES). Furthermore, we provide some comparison with a novel risk measure based on the expectiles. Theoretically, the EVaR is defined as a maximizer of the VaR through the Chernoff inequality. It gives the relation between the CDF of a distribution and its exponential moments – thus, the moment generating function arises. The fact that the entropic VaR is based on the exponential moments leads us to expect that it has to react faster to the negative scenarios.

In principle, the very concept of risk is closely related to the rare events that are placed in the distribution tails. This determines the small number of such observations in statistical data. To overcome this problem, we assume that the financial instrument we are interested in is governed by some stochastic model. Our choice falls on five fundamental models that capture the main stylized financial facts – Black-Scholes, exponential tempered stable, Heston, stochastic volatility with compound Poisson jumps (Bates), and stochastic volatility with tempered stable jumps. Based on their MGFs, we derive closed-form formulas for the EVaR. A specific technique is applied to deal with the unobservable values of the volatility that appear in Heston's model and its successors. Our suggestion is to average over all possible volatility values using its stationary gamma distribution. The EVaR for this modification is also obtained.

In addition, this study gives a strong practical flavor to the EVaR topic based on large statistical data for the S&P500 index. The results of the empirical calculations are profoundly analyzed and interpreted. The theoretical comparison between all risk measures is continued from a financial viewpoint. We have a confirmation that the entropic VaR is a risk measure that almost immediately indicates possible market turbulence.

Furthermore, the paper reveals several areas that can ignite further scientific and practical endeavours. Being the VaR-supremum, the Entropic VaR enables the investors to identify the most conservative scenarios of portfolio optimization. On the other hand, the Entropic VaR indicates to

the regulators the ultimate capital to allocate for risk mitigation purposes.

Declarations

Use of AI tools

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability statement

All data generated or analyzed during this study are included in this article.

Ethical approval (optional)

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable

Conflicts of interest

The authors declare that they have no conflict of interest.

Funding

The work was partially supported by the Centre of Excellence in Informatics and ICT under the Grant No BG16RFPR002-1.014-0018-C01, financed by the Research, Innovation and Digitalization for Smart Transformation Programme 2021-2027 and co-financed by the European Union.

Author's contributions

D.N.: Conceptualization, Methodology, Formal analysis, Software, Investigation, Writing - original draft, Writing - review & editing. T.Z.: Conceptualization, Methodology, Formal analysis, Software, Investigation, Writing - original draft, Writing - review & editing. All authors have read and agreed to the published version of the manuscript and contributed substantially and comparably to this work.

Acknowledgements

The authors would like to express sincere gratitude to the editors and to the anonymous reviewers for the helpful and constructive comments, which substantially improved the quality of this paper.

Appendix A Proofs

Proof (*Proposition 4*) Suppose first that $\alpha_1 < 1$. We have $-\lambda_1 \psi_1'(-\lambda_1) = +\infty$ because

$\lim_{z \nearrow \lambda_1} \left(1 - \frac{z}{\lambda_1}\right)^{\alpha_1 - 1} = +\infty$ and $\Gamma(-\alpha_1) < 0$. Having in mind that the rest of the terms in $h(c_{\xi}, \epsilon; \xi)$ are finite, we conclude that $a_{ETS} < c_{ETS}$. If $\alpha_1 = 1$, the same result holds since $\lim_{z \nearrow \lambda_1} \ln \left(1 - \frac{z}{\lambda_1}\right) = -\infty$. It remains to apply the second part of [Theorem 1](#).

Suppose now that $1 < \alpha_1 < 2$. Let us consider the part of $h(\lambda_1, \epsilon; \xi)$ related to $\psi_1(\cdot)$ as a function of α_1 , say $f(x)$. It and its derivative are

$$\begin{aligned} f(x) &= -\psi_1'(-x) \lambda_1 - \psi_1(-x) \\ &= c_1 x \Gamma(-x) \lambda_1^x + c_1 (1-x) \Gamma(-x) \lambda_1^x \\ &= c_1 \Gamma(-x) \lambda_1^x, \\ f'(x) &= c_1 \Gamma(-x) \lambda_1^x (\ln \lambda_1 - D(-x)), \end{aligned} \quad (46)$$

where $D(\cdot)$ is the digamma function. Remind that this is the derivative of the log-gamma function. Having in mind that it increases between minus and plus infinity in the interval $(-2, -1)$, we conclude that $f'(x) < 0$ for $x \in (1, -D^{-1}(\ln \lambda_1))$ and $f'(x) > 0$ if $x \in (-D^{-1}(\ln \lambda_1), 2)$.

Hence, $f(x)$ decreases for $x \in (1, -D^{-1}(\ln \lambda_1))$ and increases for $x \in (-D^{-1}(\ln \lambda_1), 2)$. Thus the function $f(x)$ has minimum for $x = -D^{-1}(\ln \lambda_1)$ and its value is $f(-D^{-1}(\ln \lambda_1)) = b$ (see formulation of **Proposition 4** for the value of b). We finish the proof using this behavior of $f(x)$ and the part of $h(\lambda_1, \epsilon; \xi)$ coming from $\psi_2(\cdot)$ and $\ln \epsilon$.

Appendix B Characteristic functions

The characteristic function of the tempered stable process at the moment t can be written as

$$\Psi_{TS}(u) = e^{t(\mu u + \psi_1(u) + \psi_2(u))}, \quad (47)$$

where

$$\begin{aligned} \psi_1(u) &= \begin{cases} \Gamma(-\alpha_1) \lambda_1^{\alpha_1} c_1 \left(\left(1 + \frac{iu}{\lambda_1}\right)^{\alpha_1} - 1 - \frac{iu\alpha_1}{\lambda_1} \right), & \text{if } \alpha_1 \neq 1, \\ -iuc_1 + c_1 (\lambda_1 + iu) \ln \left(1 + \frac{iu}{\lambda_1}\right), & \text{if } \alpha_1 = 1, \end{cases} \\ \psi_2(u) &= \begin{cases} \Gamma(-\alpha_2) \lambda_2^{\alpha_2} c_2 \left(\left(1 - \frac{iu}{\lambda_2}\right)^{\alpha_2} - 1 + \frac{iu\alpha_2}{\lambda_2} \right), & \text{if } \alpha_2 \neq 1, \\ iuc_2 + c_2 (\lambda_2 - iu) \ln \left(1 - \frac{iu}{\lambda_2}\right), & \text{if } \alpha_2 = 1. \end{cases} \end{aligned} \quad (48)$$

The characteristic function of the Heston's log-returns can be derived as

$$\Psi_{Heston}(t, x, v; u) = e^{C(t, u) + D(t, u)v}, \quad (49)$$

where

$$\begin{aligned} C(s, u) &= \mu uis + \frac{\xi \eta}{\theta^2} \left[(\xi - \rho \theta u i + d) s - 2 \log \left(\frac{1 - g \exp(d_j s)}{1 - g} \right) \right], \\ D(s, u) &= \frac{\xi - \rho \theta u i + d}{\theta^2} \frac{1 - \exp(ds)}{1 - g \exp(ds)}, \\ g &= \frac{\xi - \rho \theta u i + d}{\xi - \rho \theta u i - d}, \\ d &= \sqrt{(\rho \theta u i - \xi)^2 + \theta^2 (u i + u^2)}. \end{aligned} \quad (50)$$

Furthermore, the characteristic function averaged through the stationary gamma distribution (43) is

$$\begin{aligned}\Psi_{Heston}^{av}(t, x; u) &= \int_0^{\infty} \Psi_{Heston}(t, x, v; u) f_{\gamma}(v) dv \\ &= \int_0^{\infty} e^{C(t, u) + D(t, u)v} f_{\gamma}(v) dv \\ &= e^{C(t, u)} \psi_{\gamma}(-iD(t, u)),\end{aligned}\quad (51)$$

where $f_{\gamma}(\cdot)$ is the gamma PDF given by (41) and $\psi_{\gamma}(\cdot)$ is its characteristic function

$$\psi_{\gamma}(u) = \left(1 - \frac{iu}{\alpha}\right)^{-c} \equiv \left(1 - \frac{i\theta^2 u}{2\xi}\right)^{-\frac{2\xi\eta}{\theta^2}}. \quad (52)$$

If we have a jump component, then the characteristic function can be written as $\Psi(t, x, v; u) = \Psi_{Heston}(t, x, v; u) e^{t\psi_J(u)}$ no matter averaged or not. If the jump process is tempered stable, then $\psi_J(u) = i\mu u + \psi_1(u) + \psi_2(u)$. If we work under the Bates model with compound Poisson jumps, then

$$\psi_J(u) = \lambda \left(e^{-\frac{\delta^2 u^2}{2} + i\left(k - \frac{\delta^2}{2}\right)u} - 1 \right). \quad (53)$$

References

- [1] Wang, J., Wang, S., Lv, M. and Jiang, H. Forecasting VaR and ES by using deep quantile regression, GANs-based scenario generation, and heterogeneous market hypothesis. *Financial Innovation*, 10(1), 36, (2024).
- [2] Adegbola, M.D., Adegbola, A.E., Amajuoyi, P., Benjamin, L.B. and Adeusi, K.B. Quantum computing and financial risk management: A theoretical review and implications. *Computer Science & IT Research Journal*, 5(6), 1210-1220, (2024).
- [3] Dri, E., Yomi, A., Vetrivelan, M., Kuassivi, C. and Exposito, I.D. Exploring quantum-enhanced estimation of financial risk metrics with quantum RNG. *ArXiv Preprint, ArXiv:2502.02125*, (2025). [[CrossRef](#)]
- [4] Mozumder, S., Hassan, M.K. and Kabir, M.H. An evaluation of the adequacy of Lévy and extreme value tail risk estimates. *Financial Innovation*, 10, 100, (2024). [[CrossRef](#)]
- [5] Ortega-Jiménez, P., Pellerey, F., Sordo, M.A. and Suárez-Llorens, A. Probability equivalent level for CoVaR and VaR. *Insurance: Mathematics and Economics*, 115, 22-35, (2024). [[CrossRef](#)]
- [6] Tong, B., Diao, X. and Li, X. Forecasting VaRs via hybrid EVT with normal and non-normal filters: A comparative analysis from the Chinese stock market. *Pacific-Basin Finance Journal*, 83, 102271, (2024). [[CrossRef](#)]
- [7] Lyu, Y., Qin, F., Ke, R., Yang, M. and Chang, J. Forecasting the VaR of the crude oil market: A combination of mixed data sampling and extreme value theory. *Energy Economics*, 133, 107500, (2024). [[CrossRef](#)]

- [8] Qiu, Z., Lazar, E. and Nakata, K. VaR and ES forecasting via recurrent neural network-based stateful models. *International Review of Financial Analysis*, 92, 103102, (2024). [[CrossRef](#)]
- [9] Leccadito, A., Staino, A. and Toscano, P. A novel robust method for estimating the covariance matrix of financial returns with applications to risk management. *Financial Innovation*, 10, 116, (2024). [[CrossRef](#)]
- [10] Kamal, E. and Bouri, E. Green bond, stock, cryptocurrency, and commodity markets: a multiscale analysis and portfolio implications. *Financial Innovation*, 11, 100, (2025). [[CrossRef](#)]
- [11] Gulliksson, M., Mazur, S. and Oleynik, A. Minimum VaR and minimum CVaR optimal portfolios: The case of singular covariance matrix. *Results in Applied Mathematics*, 26, 100557, (2025). [[CrossRef](#)]
- [12] Dhingra, V., Sharma, A. and Gupta, S.K. A comprehensive evaluation of constrained mean-expectile portfolios with short selling. *Annals of Operations Research*, (2024). [[CrossRef](#)]
- [13] Artzner, P., Delbaen, F., Eber, J.M. and Heath, D. Coherent measures of risk. *Mathematical Finance*, 9(3), 203-228, (1999). [[CrossRef](#)]
- [14] Föllmer, H. and Schied, A. Convex measures of risk and trading constraints. *Finance and Stochastics*, 6, 429-447, (2002). [[CrossRef](#)]
- [15] Föllmer, H. and Schied, A. *Stochastic Finance: An Introduction in Discrete Time*. Walter de Gruyter: New York, (2011).
- [16] Rockafellar, R.T., Uryasev, S. and Zabarankin, M. Generalized deviations in risk analysis. *Finance and Stochastics*, 10, 51-74, (2006). [[CrossRef](#)]
- [17] Föllmer, H. and Knispel, T. Entropic risk measures: Coherence vs. convexity, model ambiguity and robust large deviations. *Stochastics and Dynamics*, 11(02n03), 333-351, (2011). [[CrossRef](#)]
- [18] Föllmer, H. and Knispel, T. Convex risk measures: Basic facts, law-invariance and beyond, asymptotics for large portfolios. In *Handbook of the Fundamentals of Financial Decision Making*, (pp. 507-554). World Scientific, (2013). [[CrossRef](#)]
- [19] Brandtner, M., Kürsten, W. and Rischau, R. Entropic risk measures and their comparative statics in portfolio selection: Coherence vs. convexity. *European Journal of Operational Research*, 264(2), 707-716, (2018). [[CrossRef](#)]
- [20] Chengli, Z. and Yan, C. Coherent risk measure based on relative entropy. *Applied Mathematics & Information Sciences*, 6(2), 233-238, (2012).
- [21] Assa, H., Morales, M. and Omidirouzi, H. On the capital allocation problem for a new coherent risk measure in collective risk theory. *Risks*, 4(3), 30, (2016). [[CrossRef](#)]
- [22] Xia, J. Optimal investment with risk controlled by weighted entropic risk measures. *SIAM Journal on Financial Mathematics*, 15(1), 54-92, (2024). [[CrossRef](#)]
- [23] Zhou, R., Liu, X., Yu, M. and Huang, K. Properties of risk measures of generalized entropy in portfolio selection. *Entropy*, 19(12), 657, (2017). [[CrossRef](#)]
- [24] Zhong, W. Portfolio optimization under entropic risk management. *Acta Mathematica Sinica, English Series*, 25, 1113-1130, (2009). [[CrossRef](#)]
- [25] Ahmadi-Javid, A. An information-theoretic approach to constructing coherent risk measures. In *Proceedings, 2011 IEEE International Symposium on Information Theory*, 2125-2127, (2011). [[CrossRef](#)]
- [26] Ahmadi-Javid, A. Entropic value-at-risk: A new coherent risk measure. *Journal of Optimization*

Theory and Applications, 155, 1105-1123, (2012). [[CrossRef](#)]

- [27] Delbaen, F. Remark on the paper "Entropic Value-at-Risk: A New Coherent Risk Measure" by Amir Ahmadi-Javid, J. Optim. Theory Appl., 155(3) (2001) 1105–1123. In *Risk and Stochastics* (pp. 151-158). World Scientific, (2019). [[CrossRef](#)]
- [28] Ahmadi-Javid, A. and Fallah-Tafti, M. Portfolio optimization with entropic value-at-risk. *European Journal of Operational Research*, 279(1), 225-241, (2019). [[CrossRef](#)]
- [29] Ahmed, D., Soleymani, F., Ullah, M.Z. and Hasan, H. Managing the risk based on entropic value-at-risk under a normal-Rayleigh distribution. *Applied Mathematics and Computation*, 402, 126129, (2021). [[CrossRef](#)]
- [30] Mishura, Y., Ralchenko, K., Zelenko, P. and Zubchenko, V. Properties of the entropic risk measure EVaR in relation to selected distributions. *Modern Stochastics: Theory and Applications*, 11(4), 373-394, (2024). [[CrossRef](#)]
- [31] Ahmadi-Javid, A. and Pichler, A. An analytical study of norms and Banach spaces induced by the entropic value-at-risk. *Mathematics and Financial Economics*, 11(4), 527-550, (2017). [[CrossRef](#)]
- [32] Axelrod, A. and Chowdhary, G. A dynamic risk form of entropic value at risk. In *Proceedings, AIAA Scitech 2019 Forum*, pp. 0392, San Diego, California, (2019, January).
- [33] Yang, J. and Qiu, W. A measure of risk and a decision-making model based on expected utility and entropy. *European Journal of Operational Research*, 164(3), 792-799, (2005). [[CrossRef](#)]
- [34] Gençay, R. and Gradojevic, N. Crash of '87 — Was it expected?: Aggregate market fears and long-range dependence. *Journal of Empirical Finance*, 17(2), 270-282, (2010). [[CrossRef](#)]
- [35] Bowden, R.J. Directional entropy and tail uncertainty, with applications to financial hazard. *Quantitative Finance*, 11(3), 437-446, (2011). [[CrossRef](#)]
- [36] Pichler, A. and Schlotter, R. Entropy based risk measures. *European Journal of Operational Research*, 285(1), 223-236, (2020). [[CrossRef](#)]
- [37] Zou, Z., Wu, Q., Xia, Z. and Hu, T. Adjusted Rényi entropic value-at-risk. *European Journal of Operational Research*, 306(1), 255-268, (2023). [[CrossRef](#)]
- [38] Jia, L. A new coherent risk measure of entropic value at risk for uncertain systems. *Journal of Uncertain Systems*, 17(01), 2350013, (2024). [[CrossRef](#)]
- [39] Chennaf, S. and Amor, J.B. Entropic value at risk to find the optimal uncertain random portfolio. *Soft Computing*, 27, 15185-15197, (2023). [[CrossRef](#)]
- [40] Liu, W., Yang, L. and Yu, B. Distributionally robust optimization based on Kernel density estimation and mean-entropic value-at-risk. *INFORMS Journal on Optimization*, 5(1), 68-91, (2023). [[CrossRef](#)]
- [41] Ardakani, O.M. Coherent measure of portfolio risk. *Finance Research Letters*, 57, 104222, (2023). [[CrossRef](#)]
- [42] Zou, Z., Xia, Z. and Hu, T. Tsallis value-at-risk: generalized entropic value-at-risk. *Probability in the Engineering and Informational Sciences*, 38(1), 1-20, (2024). [[CrossRef](#)]
- [43] Yoshioka, H. and Yoshioka, Y. Statistical evaluation of a long-memory process using the generalized entropic value-at-risk. *Environmetrics*, 35(4), e2838, (2024). [[CrossRef](#)]
- [44] Cont, R. Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance*, 1(2), 223, (2001). [[CrossRef](#)]

- [45] Black, F. and Scholes, M. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3), 637-654, (1973). [[CrossRef](#)]
- [46] Küchler, U. and Tappe, S. Exponential stock models driven by tempered stable processes. *Journal of Econometrics*, 181(1), 53-63, (2014). [[CrossRef](#)]
- [47] Rachev, S., Kim, Y., Bianchi, M. and Fabozzi, F.J. *Financial Models with Lévy Processes and Volatility Clustering*. John Wiley & Sons: New York, (2011).
- [48] Cont, R. and Tankov, P. *Financial Modeling with Jump Processes*. Chapman & Hall/CRC Press: New York, (2004).
- [49] Heston, S. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2), 327-343, (1993). [[CrossRef](#)]
- [50] Bates, D. Jumps and stochastic volatility: The exchange rate processes implicit in Deutschemark options. *Review of Financial Studies*, 9(1), 69-107, (1996). [[CrossRef](#)]
- [51] Zhevski, T.S., Kim, Y.S. and Fabozzi, F.J. Option pricing under stochastic volatility and tempered stable Lévy jumps. *International Review of Financial Analysis*, 31, 101-108, (2014). [[CrossRef](#)]
- [52] Zhevski, T.S. and Nedeltchev, D. From BASEL III to BASEL IV and beyond: Expected shortfall and expectile risk measures. *International Review of Financial Analysis*, 87, 102645, (2023). [[CrossRef](#)]
- [53] Tüysüz, Ş. and Gül, M. Investigating Türkiye's financial nexus: A wavelet coherence analysis of sovereign CDS spreads, bond yields, stock index, and FX rates. *Mathematical Modelling and Numerical Simulation with Applications*, 4(4), 395-415, (2024). [[CrossRef](#)]
- [54] Kaymak, Ö.Ö., Kaymak, Y., Mirgen, Ç., Emir, S. and Koyuncuoğlu, H.C. Credit default swaps prediction by using an FTS-ANN model. *Mathematical Modelling and Numerical Simulation with Applications*, 4(5), 187-206, (2024). [[CrossRef](#)]
- [55] Chernoff, H. A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *The Annals of Mathematical Statistics*, 23(4), 493-507, (1952).
- [56] Lukacs, E. *Characteristic Functions* (Vol. 2). Griffin: London, (1970).
- [57] Delbaen, F. Coherent risk measures on general probability spaces. In *Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann* (pp. 1-37). Heidelberg, Germany: Springer, (2002). [[CrossRef](#)]
- [58] del Baño Rollin, S., Ferreira-Castilla, A. and Utzet, F. On the density of log-spot in the Heston volatility model. *Stochastic Processes and their Applications*, 120(10), 2037-2063, (2010). [[CrossRef](#)]
- [59] Dragulescu, A.A. and Yakovenko, V.M. Probability distribution of returns in the Heston model with stochastic volatility. *Quantitative Finance*, 2(6), 443, (2002). [[CrossRef](#)]
- [60] Zhevski, T. and Nedeltchev, D. Moment generating function of the averaged log-returns in the Heston's stochastic volatility model. In *Proceedings, Comptes rendus de l'Académie bulgare des Sciences*, 78(3), 321-330, (2025). [[CrossRef](#)]
- [61] Rouah, F.D. *The Heston Model and its Extensions in Matlab and C#*. John Wiley & Sons: New Jersey, (2013).

Mathematical Modelling and Numerical Simulation with Applications (MMNSA)
(<https://mmnsa.researchcommons.org>)



Copyright: © 2026 by the authors. This work is licensed under a Creative Commons Attribution 4.0 (CC BY) International License. The authors retain ownership of the copyright for their article, but they allow anyone to download, reuse, reprint, modify, distribute, and/or copy articles in MMNSA, so long as the original authors and source are credited. To see the complete license contents, please visit (<http://creativecommons.org/licenses/by/4.0/>).

How to cite this article: Nedeltchev, D. & Zhevski, T. (2026). Measuring market risk through Entropic VaR. *Mathematical Modelling and Numerical Simulation with Applications*, 6(1), 251-278. <https://doi.org/10.53391/2791-8564.1024>